System reliabiliy estimation using conditional Monte Carlo simulation

Arne Bang Huseby
Department of Mathematics
University of Oslo
Norway
arne@math.uio.no

Abstract

The paper considers the problem of estimating system reliability using Monte Carlo simulation. By conditioning on suitable functions of the component state vector, the Monte Carlo estimate may converge much faster to the true reliability. In the paper we demonstrate how this can be done using a combination of upper and lower bounds and component counts. The method is especially well suited for estimating reliability of network systems. For such systems the method can be optimized using the well-known domination invariant. After presenting the main ideas, a few illustrating examples are included as well as comparison with other similar methods.

1. Introduction

The problem of computing the reliability of a complex multicomponent system has been studied extensively over many years. Finding exact analytical solutions is often very difficult. In general this problem is known to be NP-hard. An attractive alternative to an analytical approach is to use Monte Carlo simulations. Since, however, failure events often have very low probability, a large number of simulations is needed in order to obtain stable results. Still by conditioning on suitable functions of the component state vector the convergence can be accelerated. Variations of this idea has been explored by several authors. There are two main approaches to this: (i) conditioning on upper and lower bounds of the structure function, and (ii) conditioning on component counts. The first approach is studied in Fishman (1986a) and Fishman (1986b). See also the more recent papers by Cancela and El Khadari (1995), Cancela and El Khadari (1998) and Cancela and El Khadari (2003) where a recursive procedure is used, and Ross (1994). The second approach is used in Huseby and Naustdal (2003). For a more thorough discussion and a comparison of the two approaches we refer to Huseby et.al. (2004).

In the present paper we combine the two approaches. That is, we condition both on upper and lower bounds and on component counts. This way we are able to improve both approaches. Before we explain the ideas further, we introduce the following standard assumptions and notation. Let $\mathbf{X} = (X_1, \dots, X_n)$ be a vector of n independent bernoulli variables, interpreted as the component state vector relative to a system of n components. A component is *failed* if its state variable is 0, and *functioning* if the state variable is 1. We also introduce the *structure function* of the system, ϕ, which is also assumed to be bernoulli, and interpreted as the *system state*. If ϕ is 0, the system is *failed*, whereas if ϕ is 1, the system is *functioning*. The function ϕ is assumed to be a *nondecreasing* function of the component state vector, $\mathbf{X}$. The expected value of ϕ, denoted h, is called the *system reliability*, and is equal to $\Pr(\phi = 1)$. The goal of the analysis is to obtain an estimate for h based on a Monte Carlo simulation of the system.

The upper and lower bound approach is based on finding two simpler structure functions, ϕ_L and ϕ_U, such that $\phi_L \le \phi \le \phi_U$. We can then write:

(1.1) $h = E[\phi] = \theta_{1,1} \Pr(\phi_L = 1, \phi_U = 1) + \theta_{0,1} \Pr(\phi_L = 0, \phi_U = 1) + \theta_{0,0} \Pr(\phi_L = 0, \phi_U = 0)$

$= h_L + \theta_{0,1} (h_U - h_L)$,

where $\theta_{i,j} = E[\phi|\phi_L = i, \phi_U = j]$, $h_L = E[\phi_L]$ and $h_U = E[\phi_U]$. Assuming that h_L and h_U can be calculated analytically in polynomial time, we only need to estimate $\theta_{0,1}$. This is done by sampling from the conditional distribution of $\mathbf{X}$ given $\phi_L = 0$ and $\phi_U = 1$. If ϕ_L and ϕ_U are close approximations to ϕ, a lot can be gained by using this procedure. Several different methods for constructing upper and lower bound functions are studied in Huseby et.al. (2004) including the one given in Fishman (1986a). An efficient algorithm for sampling from the conditional distribution is also provided.

The component count approach is based on selecting a subset A of the component set, C, and conditioning on the number of functioning components within A. Thus, we introduce the following quantities:

(1.2) $S_A = \sum_{i \in A} X_i$, and $\theta_s = E[\phi|S_A = s]$, $s = 0, 1, \ldots, |A|$.

We can then write:

(1.3) $$h = E[\phi] = \sum_{s=0}^{|A|} \theta_s \Pr(S_A = s)$$

The distribution of S_A can easily be calculated in $O(|A|)$-time. Thus, in order to estimate h using this setup, we just need to estimate $\theta_0, \theta_1, \ldots, \theta_{|A|}$. This is done by sampling from the conditional distributions of $\mathbf{X}$ given S_A. In Huseby and Naustdal (2003) this is done for the case when A is the entire set of components. A special algorithm is derived for the case when all the components have equal reliability. Extending all the sampling procedures to cases where A is a subset is trivial.

In general there is no uniformly best among the two above approaches. For some systems the upper and lower bound approach yields the best results while for other systems it is better to condition on component counts. For the special case where all components have equal reliabilities, though, the second approach has a clear advantage. In this particulary case both the conditional distributions of $\mathbf{X}$ given S_A as well as the conditional expectations, $\theta_0, \theta_1, \ldots, \theta_{|A|}$, do not depend on the common component reliability. Thus, denoting the common component reliability by p, the entire reliability polynomial, $h(p)$, can be estimated simultaneously for all p using common estimates for the θ_s's.

2. A combined approach

In order to combine the two approaches discussed in the previous section, we construct the upper and lower bounds using what is called the *factoring method*. That is, we choose a subset A of the component set, C, and define ϕ_L and ϕ_U as follows:

(2.1) $\phi_L(\mathbf{X}^{C\setminus A}) = \phi(\mathbf{0}^A, \mathbf{X})$, and $\phi_U(\mathbf{X}^{C\setminus A}) = \phi(\mathbf{1}^A, \mathbf{X})$,

where $\mathbf{X}^{C\setminus A}$ denotes the subvector of $\mathbf{X}$ corresponding to the set $C\setminus A$, while $(\mathbf{0}^A, \mathbf{X})$ and $(\mathbf{1}^A, \mathbf{X})$ denotes the vectors obtained from $\mathbf{X}$ by replacing all entries corresponding to the set A by 0 or 1 respectively. As a result we obviously get that $\phi_L \le \phi \le \phi_U$. We assume that the set A is chosen so that the reliabilities of ϕ_L and ϕ_U can be calculated in polynomial time. This would be the case if e.g., both these structures are *s-p-structures*, i.e., structures which can be reduced to single components by applying series and parallel reductions. It is easy to see that A can always be chosen so that this is the case.

As in the previous section we let S_A denote the component count corresponding to the set A, i.e., the sum of X_i's for all $i \in A$. In this case, however, we introduce the following conditional expectations:

(2.2) $\theta_s = E[\phi | S_A = s, \phi_L = 0, \phi_U = 1], s = 0, 1, \ldots, |A|.$

Since the X_i's are assumed to be independent, it follows that S_A is independent of ϕ_L and ϕ_U. As a result we get that:

$$h = E[\phi] = h_L + \sum_{s=0}^{|A|} \theta_s (h_U - h_L) \Pr(S_A = s) \quad (2.3)$$

Thus, an estimate of h can be obtained by estimating $\theta_0, \theta_1, \ldots, \theta_{|A|}$. This is done by sampling from the conditional distribution of $\mathbf{X}$ given S_A, ϕ_L and ϕ_U. Using the independence of the X_i's again it follows that $\mathbf{X}^A$ is independent of ϕ_L and ϕ_U, while $\mathbf{X}^{C\setminus A}$ is independent of S_A. Hence, $\mathbf{X}^A$ can be sampled from the conditional distribution of $\mathbf{X}^A$ given S_A using the procedure derived for the component count approach. Similarly $\mathbf{X}^{C\setminus A}$ can be sampled from the conditional distribution of $\mathbf{X}^{C\setminus A}$ given ϕ_L and ϕ_U using the procedure derived for the upper and lower bound approach.

It is easy to see that the combined approach improves both the previous approaches. It should be noted, however, that the efficiency of this method depends on the choice of the set A. Intuitively, one should try to minimize the size of the set as much as possible. Moreover, one should choose A so that the upper and lower bounds are close and provide as much information about the subvector $\mathbf{X}^{C\setminus A}$ as possible. These considerations can be studied using the theory of domination. See Huseby et. al. (2004) for more details.

For the special case when all components have equal reliabilities we can sample $\mathbf{X}^A$ using the special algorithm developed for the component count approach. See Huseby and Naustdal (2003) for more details. However, it should be noted that in this case the conditional expectations, $\theta_0, \theta_1, \ldots, \theta_{|A|}$, will depend on the common component reliability. Thus, we cannot use common estimates for the θ_s's to compute the entire reliability polynomial, $h(p)$ as we could with the component count approach.

3. Applying the improved sampling method on a network system

In this section we illustrate the combined approach on a specific example: a 2-terminal undirected network system consisting of 49 nodes organized in a 7×7 grid and connected horizontally and vertically by 84 edges (42 horizontal edges and 42 vertical edges). The terminals are located in the first and the last column of the the fourth row of nodes. The nodes are assumed to function perfectly, while the edges have a common reliability p. As the set A, we select all vertical edges, except those in the first and last column. Thus, $|A| = 30$. The reliability polynomial, $h(p)$, is estimated for $p = 0.00, 0.01, \ldots, 0.99, 1.00$. For all the edges outside the set A the states are generated using common random numbers for all values of p.

As a comparison we also estimate the reliability polynomial using crude Monte Carlo simulations, again using common random numbers for all values of p. In Figure 1 the resulting simulation results are shown. Both plots contain four curves each representing 500 iterations. As is seen, the combined method produces four almost identical curves indicating a very good convergence. The crude Monte Carlo method, on the other hand, produces far less stable results. We have also compared the combined method to results using either upper and lower bounds or component counts. We did not have room for he results of these simulations in this brief paper, but both these methods were outperformed by the combined approach.

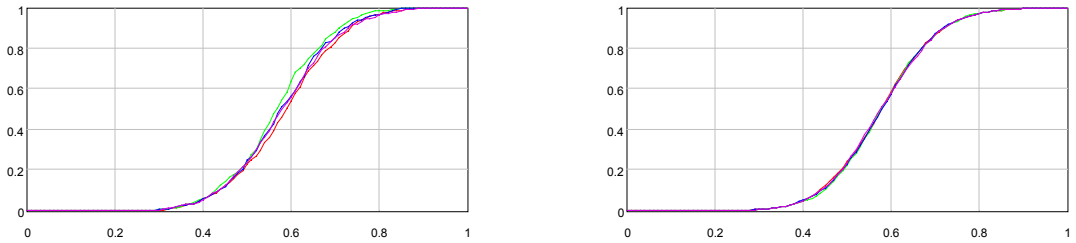

Figure 1: Reliability polynomials for a 2-terminal undirected network system estimated using crude Monte Carlo (left) and combined conditional Monte Carlo (right).

4. Conclusions

By conditioning on upper and lower bounds and component counts in Monte Carlo simulations, we are able to obtain improved results compared to conditioning either on bounds or on counts. The method produces stable estimates for large scale systems even for small or moderate sample sizes.

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